ON PARTICULAR CASES OF MOTION OF A HEAVY RIGID BODY ABOUT A FIXED POINT*

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1. Some information about the particular solutions of the problem. The problem of motion of a heavy rigid body about a fixed point under the action of the force of gravity can be reduced to finding the general or particular solutions of the following systems of differential equations

$$A\frac{dp}{dt} = (B - C)\,qr + Mg\,(y_0\gamma_3 - z_0\gamma_2)$$

$$B\frac{dq}{dt} = (C - A)\,rp + Mg\,(z_0\gamma_1 - x_0\gamma_3) \qquad (1.1)$$

$$C\frac{dr}{dt} = (A - B)\,pq + Mg\,(x_0\gamma_2 - y_0\gamma_1)$$

$$\frac{d\gamma_1}{dt} = r\gamma_2 - q\gamma_3, \qquad \frac{d\gamma_2}{dt} = p\gamma_3 - r\gamma_1, \qquad \frac{d\gamma_3}{dt} = q\gamma_1 - p\gamma_2 \qquad (1.2)$$

where p, q, r are the projections of the instantaneous angular velocity vector of rotation on the moving coordinate axes OX, OY, OZ, which are rigidly connected with the body and directed along the principal axes of the inertia ellipsoid, constructed with respect to the fixed point; A, B, C, are the principal moments of inertia with respect to the axes OX, OY, OZ; M is the mass of the body; g is the acceleration due to gravity;

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* The present paper is an abreviated exposition of the author's candidate's dissertation, "Certain necessary conditions for the existence of single-valued solutions in the problem of motion of a heavy rigid body about a fixed point", defended in 1950 at the Institute of Mechanics of the Academy of Sciences of the U.S.S.R. Conditions are added for the existence of the Grioli case (1947) for which in the dissertation only permanent rotations were obtained.

x_0, y_0, z_0 are the coordinates of the center of gravity with respect to the moving coordinate system; γ_1, γ_2, γ_3, are the direction cosines of the vertical axis OZ_1 along which the gravity force is acting.

The general solution of the systems of equations (1.1) and (1.2) depends on six arbitrary constants. Because of the relation $\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$, the number of arbitrary constants on which the functions p, q, r, γ_1, γ_2, γ_3 depend, will be equal to five.

Equations (1.1) and (1.2) do not contain time t explicitly, and their last multiplier is equal to one [32]. Therefore, for the reduction of the problem to quadratures, it is sufficient to have only four independent first integrals.

Three classical algebraic first integrals are known, namely

$$Ap^2 + Bq^2 + Cr^2 - 2Mg(x_0\gamma_1 + y_0\gamma_2 + z_0\gamma_3) = h$$
$$Ap\gamma_1 + Bq\gamma_2 + Cr\gamma_3 = k \qquad (1.3)$$
$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = l_0^2 = 1$$

The first integral is the so-called energy integral, the second integral expresses the law of conservation of the angular momentum about the vertical and the third integral expresses a property of the direction cosines.

A fourth algebraic first integral for arbitrary values of the coefficients of equations (1.1) and (1.2) has not been found. Under certain restrictions concerning the location of the center of gravity and the values of A, B, C, such a fourth integral can be found.

Up to the second half of the nineteenth century the following cases of integrability were found and investigated.

1. Case of Euler and Poinsot, when $x_0 = y_0 = z_0 = 0$ and the fourth algebraic first integral is

$$A^2p^2 + B^2q^2 + C^2r^2 = \text{const}$$

2. Case of Lagrange and Poisson, when $A = B$, $x_0 = y_0 = 0$, $z_0 \neq 0$ and the fourth algebraic first integral is

$$r = \text{const}$$

3. Case of complete kinetic symmetry, when $A = B = C$ and the fourth algebraic first integral is

$$x_0p + y_0q + z_0r = \text{const}$$

In 1888 appeared a remarkable investigation by Kovalevskaia [1], in which a new case of integrability was discovered and investigated. In this case, which bears her name, and which is realized when

$$A = B = 2C, \qquad z_0 = 0$$

(by a rotation of axes in the *XY*-plane we can make $y_0 = 0$) there exists a fourth algebraic first integral

$$[C(p^2 - q^2) + Mgx_0\gamma_1]^2 + [2Cpq - Mgx_0\gamma_2]^2 = \text{const}$$

This memoir of Kovalevskaia's stimulated a large number of investigations referring (1) to the question of finding new particular solutions of the general problem, (2) to the question of finding particular solutions of the new case, and (3) to the clarifying of the geometrical picture and the details of motion in the cases of the known particular solutions.

A number of questions connected with the geometric representation of the various cases of motion and with the question of finding particular solutions of the general problem were solved by Zhukovskii, Liapunov, Chaplygin, Steklov, Mlodzeevskii *et al.*

Kovalevskaia raised the problem of finding all the cases when the general solution of the systems (1.1) and (1.2) can be expressed in terms of single-valued functions of t, these functions having no other singularities than poles for all finite values of t, t being a complex variable.

These functions can be expanded in series of the form

$$\begin{aligned} p &= \frac{1}{t^{n_1}}(p_0 + p_1 t + p_2 t^2 + \ldots), & \gamma_1 &= \frac{1}{t^{m_1}}(\gamma_0' + \gamma_1' t + \gamma_2' t^2 + \ldots) \\ q &= \frac{1}{t^{n_2}}(q_0 + q_1 t + q_2 t^2 + \ldots), & \gamma_2 &= \frac{1}{t^{m_2}}(\gamma_0'' + \gamma_1'' t + \gamma_2'' t^2 + \ldots) \\ r &= \frac{1}{t^{n_3}}(r_0 + r_1 t + r_2 t^2 + \ldots), & \gamma_3 &= \frac{1}{t^{m_3}}(\gamma_0''' + \gamma_1''' t + \gamma_2''' t^2 + \ldots) \end{aligned} \qquad (1.4)$$

where n_1, n_2, n_3, m_1, m_2 and m_3 are positive integers.

In order that in the general case the systems (1.1) and (1.2) be integrable by series of the form (1.4), which contain five arbitrary constants, it is necessary that the coefficients of these series satisfy definite conditions. One such condition gave the new case of integrability considered by Kovalevskaia.

Let us quote two theorems which complete the problem of finding all cases for which single-valued solutions for arbitrary initial conditions exist (i.e. the problem of finding general solutions).

Theorem of Kovalevskaia [2]. In the general case, equations (1.1) and (1.2) do not admit single-valued solutions containing five arbitrary constants and having on the whole plane of the variable t no singular points other than poles. The exceptional cases are:

$$(1)\quad A = B = C, \qquad (3)\quad A = B,\ x_0 = y_0 = 0$$

$$(2)\quad x_0 = y_0 = z_0 = 0 \qquad (4)\quad A = B = 2C,\ z_0 = 0 \qquad (1.5)$$

Theorem of Liapunov [11]. Of all the cases when the constants A, B, C, x_0, y_0 and z_0 are real and A, B and C different from zero, the above cases (1.5) are the only ones when the functions p, q, r, γ_1, γ_2 and γ_3, determined by the equations (1.1) and (1.2), are single-valued for any initial values of these functions.

Considering series of the form (1.4) Kovalevskaia takes for the constants n_i and m_i the following values: $n_i = 1$, $m_i = 2$ $(i = 1, 2, 3)$, leaving unconsidered the question whether this system of values is unique or not.

Nekrasov and Appel'rot [5,6], investigating the exponents n_i, m_i, and Liapunov, establishing the above theorem, pointed out a particular case, that of the so-called loxodromic pendulum, overlooked by Kovalevskaia.

The method proposed by Kovalevskaia in the problem of motion of a rigid body was not developed further for this problem.

No-one succeeded in determining the existence of particular solutions by Kovalevskaia's method, except in the case of the loxodromic pendulum.

All the particular solutions rediscovered were found by a skilful use of the differential equations under consideration or by the investigation of certain particular properties of the fourth algebraic first integral. A general method, similar to Kovalevskaia's, for determining particular solutions, has not been found.

Appel'rot [5] established the following theorem. In the case of three unequal moments of inertia there are neither general integrals nor particular integrals of the differential equations of motion of a heavy rigid body, having for p, q, and r poles of order higher than one and for γ_1, γ_2 and γ_3 poles of order higher than two. If, however, $A = B$, $y_0 = 0$, $A \neq C$, $x_0 \neq 0$, there can exist particular integrals having for p and q poles of order three.

Appel'rot did not give an example of a particular solution confirming his theorem.

In the present paper this theorem is confirmed for the case of Goriachev and Chaplygin.

From the theorem of Liapunov it follows that besides (1.5) there cannot be new cases when single-valued general solutions can be found (i.e.

single-valued solutions for arbitrary initial values). If restrictions are imposed on the constant h in the energy integral, on the constant k in the angular momentum integral about the vertical, and on the values of A, B, C, x_0, y_0 and z_0, then, in particular cases, a solution of the problem is possible. In the general form, however, the nature of such restrictions has not been investigated by anyone.

Such a problem was not raised and did not follow from Kovalevskaia's method.

In an unpublished manuscript Chaplygin [27] made an attempt to obtain the integrable cases by a unique method.

The present paper shows that Kovalevskaia's method can also be applied to find particular cases of integrability of the equations of motion.

Such a method permits only the necessary conditions for the existence of single-valued particular solutions to be found. To verify the sufficiency of the similar conditions, it is necessary to show that equations (1.1) and (1.2), under the derived conditions, can be integrated in terms of single-valued functions of time, or that a fourth algebraic integral, besides the generally known integrals (1.3), can be found.

Chaplygin [19] showed that "the problem considered does not admit a particular linear integral in cases other than those so far known".

Since Kovalevskaia's investigation, so far as we know, only the following basic cases of integrability and particular solutions of the systems (1.1) and (1.2) have been found and studied. We leave out of account the particular cases of the already known solutions and the various additions and modifications of the conditions themselves as well as of the particular solutions obtained (such as, for example, the particular cases of the Kovalevskaia integral, motions similar to the pendulum and other very simple motions).

1. Loxodromic pendulum. This case was found by Hess [3] in 1890 and rediscovered in 1892 by Nekrasov [6] and Appel'rot [5]. Further, it was investigated by Nekrasov [7,8,15], Mlodzeevskii [9], Zhukovskii[4,21], Chaplygin [14] *et al.* In this case under the conditions

$$y_0 = 0, \qquad A(B-C)x_0^2 - C(A-B)z_0^2 = 0$$

a fourth particular first integral exists in the algebraic form

$$Ax_0p + Cz_0r = 0$$

2. Case of Bobylev and Steklov (1893). This case was found simultaeously by Bobylev [17] and Steklov [18]. Under the conditions

$$2A = C, \ x_0 = y_0 = 0, \ q = 0,$$

a fourth particular integral $r = r_0$ exists (Bobylev) or, under the conditions $2A = B$, $x_0 = z_0 = 0$, $r = 0$, there exists the particular integral $q = q_0$ (Steklov).

3. Constant (permanent) rotations. This case was discovered in 1894 by Mlodzeevskii and Staude. The investigations of Mlodzeevskii [13] concerning this problem were already completed when the paper by Staude [12] appeared, in which one case referring to permanent rotations, namely when the axis of rotation is vertical, was examined. Mlodzeevskii stated the problem in a broader form.

4. Second case of Steklov (1899) [20]. Under the conditions $B > A > 2C$, $y_0 = z_0 = 0$ there exist particular joint solutions

$$\gamma_2 = \frac{(C-A)(B-A)}{Q(2C-A)}pq, \qquad \gamma_3 = \frac{(C-A)(B-A)}{Q(2B-A)}pr$$

Here, and in what follows, we have denoted $Q = Mgx_0$.

In the present paper it is shown that in this case we shall have

$$k = 0, \qquad h = \frac{Q(A^2 - 2AB - 2AC + 2BC)}{(A-B)(C-A)}$$

The solution contains a single arbitrary constant t_0.

5. Case of Goriachev and Chaplygin (1899). This case was discovered by Goriachev [23], who gave a solution which contains three arbitrary constants, and by Chaplygin [24], who obtained a solution containing four arbitrary constants. The Goriachev solution is a special case of that of Chaplygin. Sretenskii [31] investigated the motions which arise when the body is rotating with a very large velocity about the principal axis of inertia through the center of gravity. His method was later used by Arkhangel'skii [34].

In this case under the conditions

$$A = B = 4C, \qquad y_0 = z_0 = 0, \qquad k = 0$$

a fourth particular algebraic first integral

$$r(p^2 + q^2) + \frac{Q}{C}p\gamma_3 = l$$

exists, where l is an arbitrary constant.

6. Second case of Goriachev (1899)* [22]. Under the conditions

$$AC = 8(A-2B)(B-C), \qquad y_0 = z_0 = 0$$

* It is more convenient to call this case his second, though Goriachev gave the preceding one in November, while this one was in August.

particular joint integrals

$$2Q\gamma_2 = (4B - 3A)\, pq, \qquad Q\gamma_3 = (\lambda + \mu p^2)\, pr$$

exist, where

$$\lambda = \frac{(3A - 4B)(2B - C)(2B - 3C)}{BC}, \qquad \mu = \frac{\lambda A (3A - 4B)(4B - 3C)(5C - 4B)}{32QBC(B - C)}$$

In the present paper it is shown that in this case we shall have

$$k = 0, \qquad h = \frac{4Q(A - 2B)(9A^2 - 56AB + 64B^2)}{(3A - 4B)(15A^2 - 64AB + 64B^2)}$$

The solution contains a single arbitrary constant t_0.

7. *Second case of Chaplygin (1904)* [26]. Under the conditions

$$0.6 > \frac{C}{A} > 0.5965, \quad 1.5 < \frac{B}{A} < 1.5965, \quad y_0 = z_0 = 0$$

or under the following restrictions for the principal central moments of inertia L, M and N, namely

$$M > L > N, \qquad 0.9 < \frac{M - N}{L} < 1$$

particular algebraic integrals

$$Q\gamma_2 = (\alpha + \lambda p^{-\frac{4}{3}})\, pq, \qquad Q\gamma_3 = (\beta + \mu p^{-\frac{4}{3}})\, pr$$

exist, where

$$\alpha = \frac{(B - A)(C - A)}{2C - A}, \qquad \beta = \frac{(B - A)(C - A)}{2B - A}$$

$$\lambda = \frac{C(3A - 2B)}{2C - A}\, s, \qquad \mu = \frac{B(3A - 2C)}{2B - A}\, s$$

and s is determined by the condition

$$A^3(2B + 2C - 3A)\, s^3 = \frac{4(2B - A)^2(2C - A)^2 Q^2}{9(3A - 2B)(3A - 2C)}$$

The solution contains a single arbitrary constant t_0.

8. *Case of Kovalevski (1907)* [28]. Under the conditions

$$A = \frac{18B(B - C)}{9B - 10C} \quad \text{or} \quad AC = 9(A - 2B)(B - C), \qquad y_0 = z_0 = 0$$

particular algebraic integrals

$$\gamma_2 = -\frac{C}{2Q}\left[\beta_1 + 2\left(\beta_2 - \frac{A - B}{C}\right) p\right] q$$

$$\gamma_3 = \frac{B}{2Q}\left[\alpha_1'\beta_1 + 2\left(\alpha_2 - \frac{C - A}{B}\right) p + 3\alpha_3'\beta_1^{-1}p^2\right] r$$

$$q^2 = P_3(p), \qquad r^2 = P_2(p), \qquad \gamma_1 = P_3(p)$$

exist, where $P_s(p)$ is a polynomial of degree s with respect to p,

$$\alpha_2 = \frac{(2C - 3B)(81B^2 - 156BC + 61C^2)}{BC(9B - 10C)}, \qquad \beta_2 = -\frac{2B}{9B - 10C}$$

and β_1 α_1' and α_3' are determined by a system of algebraic equations with coefficients depending on A, B and C. There is no need to write these equations down. The solution of the problem contains a single arbitrary constant t_0.

9. *Case of Grioli (1947)* [30]. This case represents regular precession about a nonvertical axis. Under the conditions

$$y_0 = 0, \quad (B - C)x_0^2 - (A - B)z_0^2 = 0$$

joint particular integrals

$$p^2 + q^2 + r^2 = \text{const}, \quad x_0 p + z_0 r = \text{const}$$

exist.

In all these particular cases the restrictions imposed on the values of h, k, A, B, C, x_0, y_0 and z_0 are obtained by various methods, without applying Kovalevskaia's method.

The purpose of the present paper is not the study and analysis (a) of the various modifications and additions of the basic particular cases of integrability obtained or (b) of the various geometric and analytic methods applied to the above problem. Nor is the problem of the stability of motion considered. In this connection, no analysis of the literature on the motion of a heavy rigid body about a fixed point, which is not directly related to the problems considered in the present paper, is given.

2. On the differential equations of the problem containing the arbitrary constants of the classical first integrals. Let us represent the equations of motion of a heavy rigid body about a fixed point in a form different from (1.1) and (1.2), namely in the form indicated in the paper by Hess [3]. Let a quadratic form and $n - 1$ linear forms with respect to the variables $x_1, \ldots, x_n$ with coefficients $\alpha_1, \ldots, \alpha_n, \gamma_1, \ldots, \gamma_n$ be given in the form

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 + \ldots + x_n^2 &= l \\ \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \ldots + \alpha_n x_n &= \alpha \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \gamma_1 x_1 + \gamma_2 x_2 + \gamma_3 x_3 + \ldots + \gamma_n x_n &= \gamma \end{aligned} \qquad (2.1)$$

Denote

$$W = \begin{vmatrix} x_1 & x_2 & \dots & x_n \\ \alpha_1 & \alpha_2 & \dots & \alpha_n \\ \dots & \dots & \dots & \dots \\ \gamma_1 & \gamma_2 & \dots & \gamma_n \end{vmatrix}, \qquad H = W^2 = \begin{vmatrix} l & \alpha & \beta & \dots & \gamma \\ \alpha & (\alpha\alpha) & (\alpha\beta) & \dots & (\alpha\gamma) \\ \beta & (\beta\alpha) & (\beta\beta) & \dots & (\beta\gamma) \\ \dots & \dots & \dots & \dots & \dots \\ \gamma & (\gamma\alpha) & (\gamma\beta) & \dots & (\gamma\gamma) \end{vmatrix}$$

where

$$(\alpha\alpha) = \alpha_1^2 + \dots + \alpha_n^2, \quad (\alpha\beta) = \alpha_1\beta_1 + \dots + \alpha_n\beta_n$$

. . . .

Differentiating H with respect to x_i and taking into account (2.1) we obtain

$$2\frac{\partial H}{\partial l} x_i + \frac{\partial H}{\partial \alpha}\alpha_i + \frac{\partial H}{\partial \beta}\beta_i + \dots + \frac{\partial H}{\partial \gamma}\gamma_i = 2W\frac{\partial W}{\partial x_i} \qquad (i = 1, \dots, n) \quad (2.2)$$

The minors of the determinant W, corresponding to the elements x_i, are $W_i = \partial W / \partial x_i$. Therefore, we have the following identities:

$$W = \sum \frac{\partial W}{\partial x_i} x_i, \quad \sum \frac{\partial W}{\partial x_i}\alpha_i = 0, \dots, \sum \frac{\partial W}{\partial x_i}\gamma_i = 0$$

Multiplying equations (2.2) by x_i and adding, then by a_i and adding, and so on, we obtain the equations

$$\begin{aligned} &2\frac{\partial H}{\partial l} l + \frac{\partial H}{\partial \alpha}\alpha + \dots + \frac{\partial H}{\partial \gamma}\gamma = 2W^2 = 2H \\ &2\frac{\partial H}{\partial l}\alpha + \frac{\partial H}{\partial \alpha}(\alpha\alpha) + \dots + \frac{\partial H}{\partial \gamma}(\alpha\gamma) = 0 \\ &2\frac{\partial H}{\partial l}\beta + \frac{\partial H}{\partial \alpha}(\beta\alpha) + \dots + \frac{\partial H}{\partial \gamma}(\beta\gamma) = 0 \\ &\dots\dots\dots\dots\dots\dots \end{aligned} \qquad (2.3)$$

Denote the minors of the determinant H, corresponding to the elements of the first row, by the same letter with a subscript, and the minors, corresponding to the elements of the first column, in addition, by a prime above. Expanding H according to the elements of the first row and the first column, we obtain

$$H = H_l l + H_\alpha \alpha + \dots + H_\gamma \gamma, \quad H = H_l l + H_\alpha' \alpha + \dots + H_\gamma' \gamma \quad (2.4)$$

Because of the symmetry of the determinant H with respect to the diagonal we have $H_\alpha = H_\alpha'$, ..., $H_\gamma = H_\gamma'$. Adding term-by-term the equalities (2.4) and comparing them with (2.3), we obtain*

$$\frac{\partial H}{\partial l} = H_l, \qquad \frac{\partial H}{\partial \alpha} = 2H_\alpha, \dots, \quad \frac{\partial H}{\partial \gamma} = 2H_\gamma$$

* Combescure, Sur quelques systèmes particuliers d'équations différentielles. *J. Reine Angew. Math.* Vol. 80, pp. 33-51 (1875).

Due to (2.2) the variables x_i $(i = 1, 2, \ldots, n)$ can be expressed explicitly in terms of the forms (2.1) and their coefficients in the following form

$$H_l x_i = \sqrt{H}\ W_i - H_\alpha \alpha_i - H_\beta \beta_i - \ldots - H_\gamma \gamma_i \tag{2.5}$$

With reference to a heavy rigid body rotating about a fixed point, four vectors, intersecting at the point of support, can be determined at any instant t: the unit vector $\mathbf{z}_1{}^0$ along the fixed axis coinciding with the line of action of the gravity force, the vector of the principal kinetic momentum $\mathbf{G}$, the vector of the instantaneous rotation and the unit vector $\mathbf{n}^0$ of the position vector of the center of gravity.

The scalar products of these vectors are

$$\begin{aligned}
&1. \quad \nu = (\mathbf{G}\mathbf{G}) = A^2p^2 + B^2q^2 + C^2r^2\\
&2. \quad \rho = (\mathbf{G}\mathbf{n}^0) = \frac{1}{R_0}(Ax_0p + By_0q + Cz_0r)\\
&3. \quad \tau = (\mathbf{G}\boldsymbol{\omega}) = Ap^2 + Bq^2 + Cr^2\\
&4. \quad \sigma = (\boldsymbol{\omega}\mathbf{n}^0) = \frac{1}{R_0}(x_0p + y_0q + z_0r)\\
&5. \quad \theta = (\boldsymbol{\omega}\mathbf{z}_1{}^0) = p\gamma_1 + q\gamma_2 + r\gamma_3\\
&6. \quad w = (\boldsymbol{\omega}\boldsymbol{\omega}) = p^2 + q^2 + r^2\\
&7. \quad l_0 = (\mathbf{z}_1{}^0\mathbf{z}_1{}^0) = \gamma_1{}^2 + \gamma_2{}^2 + \gamma_3{}^2 = 1\\
&8. \quad k = (\mathbf{G}\mathbf{z}_1{}^0) = Ap\gamma_1 + Bq\gamma_2 + Cr\gamma_3\\
&9. \quad \mu = (\mathbf{n}^0\mathbf{z}_1{}^0) = \frac{1}{R_0}(x_0\gamma_1 + y_0\gamma_2 + z_0\gamma_3)\\
&10. \quad n^{0^2} = (\mathbf{n}^0\mathbf{n}^0) = \frac{1}{R_0{}^2}(x_0{}^2 + y_0{}^2 + z_0{}^2) = 1
\end{aligned} \tag{2.6}$$

The quantities l_0, k and n_0 are constants while the quantity τ, which represents twice the kinetic energy, is connected with μ in the following way:

$$\mu = \frac{\tau - h}{2MgR_0} \qquad (h = \text{const}) \tag{2.7}$$

In this way six new variables can be introduced. For our purpose take three of them, namely ν, ρ and μ. Instead of μ also τ can be taken.

The total derivatives of these quantities with respect to time, taking into account the Euler and Poisson equations (1.1) and (1.2), are

$$\frac{d\nu}{dt} = 2Mg\begin{vmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ Ap & Bq & Cr \\ x_0 & y_0 & z_0 \end{vmatrix}, \qquad \frac{d\rho}{dt} = \frac{1}{R_0}\begin{vmatrix} x_0 & y_0 & z_0 \\ Ap & Bq & Cr \\ p & q & r \end{vmatrix}$$
$$\frac{d\mu}{dt} = \frac{1}{R_0}\begin{vmatrix} x_0 & y_0 & z_0 \\ \gamma_1 & \gamma_2 & \gamma_3 \\ p & q & r \end{vmatrix} \tag{2.8}$$

Taking the 7th, 8th and 9th product from (2.6) for the forms of the type (2.1), we obtain, in conformity with (2.5),

$$
\begin{aligned}
H_0\gamma_1 &= \sqrt{H}W_{\gamma_1} - H_k Ap - H_\mu \frac{x_0}{R_0} \\
H_0\gamma_2 &= \sqrt{H}W_{\gamma_2} - H_k Bq - H_\mu \frac{y_0}{R_0} \\
H_0\gamma_3 &= \sqrt{H}W_{\gamma_3} - H_k Cr - H_\mu \frac{z_0}{R_0}
\end{aligned}
\tag{2.9}
$$

Here

$$
H = W^2 = \begin{vmatrix} l_0 & k & \mu \\ k & \nu & \rho \\ \mu & \rho & 1 \end{vmatrix}, \qquad W = \sqrt{H} = \frac{1}{R_0}\begin{vmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ Ap & Bq & Cr \\ x_0 & y_0 & z_0 \end{vmatrix}
$$

$$
\begin{aligned}
H_0 &= \nu - \rho^2, & W_{\gamma_1} &= \frac{1}{R_0}(Bqz_0 - Cry_0) \\
H_k &= \rho\mu - k, & W_{\gamma_2} &= \frac{1}{R_0}(Crx_0 - Apz_0) \\
H_\mu &= k\rho - \nu\mu, & W_{\gamma_3} &= \frac{1}{R_0}(Apy_0 - Bqx_0)
\end{aligned}
$$

Then from (2.8) and (2.9), taking into account (2.6), we obtain the following basic system of differential equations in the Hess form:

$$
\left(\frac{1}{2MgR_0}\frac{d\nu}{dt}\right)^2 = \begin{vmatrix} l_0 & k & \mu \\ k & \nu & \rho \\ \mu & \rho & 1 \end{vmatrix}, \qquad \left(\frac{d\rho}{dt}\right)^2 = \begin{vmatrix} 1 & \rho & \sigma \\ \rho & \nu & \tau \\ \sigma & \tau & w \end{vmatrix}
$$

$$
(\nu - \rho^2)\frac{d\mu}{dt} = (\tau - \sigma\rho)\frac{1}{2MgR_0}\frac{d\nu}{dt} + (k - \rho\mu)\frac{d\rho}{dt} \tag{2.10}
$$

The solution of the Hess equations gives the solution of the Buler equations, except in the cases when from the expressions (2.6) for ν, ρ, μ and (2.7) the quantities p, q and r cannot be obtained.

In 1903 Schiff [25] proposed a modification of the equations (1.1) and (1.2), reducing them to a new form similar to that already considered, taking for the new variables the expressions*

$$
\frac{1}{2Mg}(Ap^2 + Bq^2 + Cr^2)
$$

$$
\frac{1}{2(Mg)^2}(A^2p^2 + B^2q^2 + C^2r^2)
$$

$$
\frac{1}{Mg}(Ax_0p + By_0q + Cz_0r)
$$

* Schiff takes the last expression with the + sign, since he transforms the Buler equations, written down for the case when the OZ_1-axis is directed in the direction opposite to that of the force of gravity.

Assuming that the last two expressions can be constant, and giving various initial conditions, under these conditions Schiff unified several particular cases (already known). He did not consider whether the parameters and the initial conditions of the problem can be selected in such a way that these cases can be actually realized.

3. Cases of integrability, when radicals in the right-hand sides of the differential equations of the departure are absent. The basic differential equations are given in such a form that besides the considered variables ν, ρ, τ, two of which enter, in addition, in the form of derivatives, their right-hand sides contain the quantities r and w, which are expressed explicitly only in terms of the old variables.

From equalities (2.6) the following obvious relations follow:

$$\begin{aligned}
\nu - A\tau &= B(B-A)q^2 + C(C-A)r^2\\
\nu - B\tau &= A(A-B)p^2 + C(C-B)r^2\\
\nu - C\tau &= A(A-C)p^2 + B(B-C)q^2\\
\tau - Aw &= (B-A)q^2 + (C-A)r^2\\
\tau - Bw &= (A-B)p^2 + (C-B)r^2\\
\tau - Cw &= (A-C)p^2 + (B-C)q^2\\
R_0(\rho - A\sigma) &= (B-A)y_0q + (C-A)z_0r\\
R_0(\rho - B\sigma) &= (A-B)x_0p + (C-B)z_0r\\
R_0(\rho - C\sigma) &= (A-C)x_0p + (B-C)y_0q
\end{aligned} \qquad (3.1)$$

Let us express the quantities σ, w in terms of the variables ν, ρ and For this purpose eliminate twice in pairs from the first three relations of (2.6) any two variables (for example, p, q and r, q). After elimination, find the roots of the two algebraic polynomials in one of the two old variables (r and p), and substitute them into the corresponding last six relations of (3.1).

Substituting the expression for p obtained from the second relation of (2.6) into the first and third, we obtain two quadratic equations with respect to q with coefficients depending on r, ν, ρ and τ. Eliminating q from these equations, i.e. putting the resultant equal to zero, we obtain

$$\begin{aligned}
\{C[A(B-C)x_0^2 + B(A-C)y_0^2 + C(B-A)z_0^2]r^2 - 2C(B-A)R_0z_0\rho r +\\
+ (B-A)R_0^2\rho^2 + Ax_0^2(\nu - B\tau) + By_0^2(\nu - A\tau)\}^2 -\\
- 4B(B-A)y_0^2(Cz_0r - R_0\rho)^2[C(A-C)r^2 + \nu - A\tau] = 0
\end{aligned} \qquad (3.2)$$

In exactly the same way, substituting the expression for r obtained from the second of the relations (2.6) into the first and third relation, and eliminating q from the two quadratic equations so obtained, we get

$$\{A[A(B-C)x^2_0-B(A-C)y_0^2+C(B-A)z_0^2]p^2-2A(B-C)R_0x_0\rho p+ \\ +(B-C)R_0^2\rho^2+By_0^2(\nu-C\tau)+Cz_0^2(\nu-B\tau)\}^2+ \\ +4B(B-C)y_0^2(Ax_0p-R_0\rho)^2[A(A-C)p^2-\nu+C\tau]=0 \qquad (3.3)$$

If the roots of the polynomials (3.2) and (3.3) are substituted into the fifth and eighth relation of (3.1), then explicit expressions of σ and w in terms of ν, ρ and τ are obtained.

In general, such a transformation has a cumbersome form and requires particular consideration. We will introduce the following notations:

$$P=(A-B)(B-C)(C-A)$$
$$\Lambda=(B-C)x_0^2-(A-B)z_0^2$$
$$U=A(B-C)x_0^2+C(A-B)z_0^2$$
$$V=A(B-C)x_0^2-C(A-B)z_0^2$$

Then, equations (3.2) and (3.3) become quadratic and have simple expressions or the roots under the assumption that $y_0 = 0$.

For this case, and under the assumptions that $V\neq 0$ and that in the expressions for the roots of the polynomials (3.2) and (3.3) the radicals are taken with $\pm$ signs, the expressions for σ and w in terms of ν, ρ and τ are obtained in the form

$$\sqrt{AC}VR_0\sigma=\sqrt{AC}\Lambda R_0\rho\pm(A-C)x_0z_0\sqrt{(A-B)(B-C)R_0\rho^2-V(\nu-B\tau)} \qquad (3.4)$$

$$ABCV^2w=ACV^2\tau-PUR_0^2\rho^2-[A^2(B-C)x_0^2-C^2(A-B)z_0^2]V(\nu-B\tau)\pm \\ \pm 2\sqrt{AC}PR_0x_0z_0\rho\sqrt{(A-B)(B-C)R_0^2\rho^2-V(\nu-B\tau)} \qquad (3.5)$$

If $V=0$, then the following expressions are obtained for σ and w in terms of the variables ν, ρ and τ, assuming that $\rho\neq 0$ and $\nu\neq B\tau$:

$$\sigma=\frac{A+C}{2AC}\rho-\frac{(A-C)z_0^2(\nu-B\tau)}{2A(B-C)R_0^2\rho} \qquad (3.6)$$

$$Bw=\frac{(A-B)R_0^2}{A(A-C)x_0^2}(AC\sigma^2-\rho^2)+\tau= \qquad (3.7)$$

$$=\frac{A-B}{4A^2x_0^2}\left[\frac{(A-C)R_0^2}{C}\rho^2-\frac{2(A+C)z_0^2}{B-C}(\nu-B\tau)+\frac{C(A-C)z_0^4(\nu-B\tau)^2}{(B-C)^2R_0^2\rho^2}\right]+\tau$$

Substituting the expressions obtained for σ and w into equations

(2.10), we obtain three basic differential equations for the three variables ν, ρ and τ, the independent variable t being not explicitly contained in these equations.

Let us remark that the general solutions and certain particular solutions of the problem exist, either when the radical is absent in the right-hand sides of the basic differential equations (2.10), i.e. in the functions which give σ and w in terms of ν, ρ and τ, or when the indicated functions themselves become constant.

This is possible if one or several conditions are satisfied (the condition $y_0 = 0$, introduced earlier, of course, remains to be satisfied):

1. Either $V = 0$,

or for $V \neq 0$:

$$2.\ x_0 = 0. \quad 3.\ z_0 = 0. \quad 4.\ A = C. \quad 5.\ B = C. \quad 6.\ A = B \tag{3.8}$$

$$7.\ (A - B)(B - C) R_0^2 \rho^2 - V(\nu - B\tau) = \text{const}$$

$$8.\ \sigma = \sigma_0 (\text{const}), \quad w = w_0 (\text{const})$$

The second and third conditions of (3.8) taken together contain the case of Euler and Poinsot. Here the indefinite case must still be investigated. The second and sixth conditions together give the case of Lagrange and Poisson. The fourth, fifth and sixth conditions together give the case of complete kinetic symmetry. The third and sixth conditions taken together lead, in what follows, to the case of Kovalevskaia.

Consider the derivation of the particular cases of integrability of the equations. If necessary we shall pass to the old variables, in which the known cases of integrability have been considered.

If the first condition of (3.8) is satisfied, then σ and w are given in terms of ν, ρ and τ by the formulas (3.6) and (3.7). These formulas are obtained by elimination of p and r from the formulas (3.2), (3.3) and the fifth or the eighth relation of (3.1), under the assumption that $\rho \neq 0$ and $\nu \neq B\tau$.

For $y_0 = 0$, $V = 0$, equations (3.2) and (3.3) are satisfied for arbitrary p and r, if the new variables are related by the relations $\rho = 0$, $\nu = B\tau$. One of them can be taken as a particular integral, for example, $\rho = 0$. We then obtain the case of the loxodromic pendulum.

This case was obtained by Hess [3], and later, using Kovalevskaia's method by Appel'rot [5] and Nekrasov [6].

In a number of papers Nekrasov [6,7,8,15] analytically investigated the motion.

The peculiarity of this case consists in the fact that the solution of the problem reduces itself, in general, not to single-valued functions of time but to many-valued ones, and only under certain supplementary conditions can the solution be reduced to single-valued functions of time. The conditions for the existence of asymptotic periodic motions in this problem were considered by Mlodzeevskii and Nekrasov [9].

Zhukovskii [4], with his innate skill, gave a geometrical solution of the above pendulum.

Chaplygin [14] indicated the disposition of the points of support in the rigid body for the motion to be realized.

Consider the third and seventh condition of (3.8) which assume the form

$$(B-C)x_0^2[(A-B)\rho^2 - A(\nu - B\tau)] = \text{const} \tag{3.9}$$

or in the old variables $AC(B-C)x_0^2r^2$ = const. For the particular value zero of the constant when $B = C$, we obtain the case of Lagrange and Poisson. If $x_0 = 0$, we have the case of Euler and Poinsot. Excluding these cases, we obtain that the above condition is equivalent to $r = 0$.

Here the new variables in terms of the old variables are expressed as follows:

$$\nu = A^2p^2 + B^2q^2, \qquad \rho = Ap, \qquad \tau = Ap^2 + Bq^2 \tag{3.10}$$
$$\sigma = p, \qquad w = p^2 + q^2, \qquad \mu = \gamma_1$$

The second differential equation of (3.10), written in the form

$$\left(\frac{d\rho}{dt}\right)^2 = (w-\sigma^2)(\nu-\rho^2) - (\tau - \rho\sigma)^2$$

and taking into account the relation (3.10), gives $\rho = \rho_0$ (const).

The third differential equation of (2.10) is satisfied as a consequence of the relation (3.10), the condition $\rho = \rho_0$, the equality (2.7) and the second of the equalities (3.1).

The first differential equation assumes the form

$$\left(\frac{1}{2Q}\frac{d\nu}{dt}\right)^2 = (\nu - \rho_0^2)(1-\mu^2) - (k - \rho_0\mu)^2 \tag{3.11}$$

where ν and μ, owing to the relations (3.10) and (2.7), are connected by the relation

$$\mu = \frac{1}{2QB}\left(\nu - \frac{A-B}{A}\rho_0^2 - Bh\right)$$

For the conditions introduced, $y_0 = z_0 = r = 0$, $p = \rho_0/A$ = const, the third equation of Euler gives

$$(A-B)\,p_0 q + Q\gamma_2 = 0$$

The first two integrals of (1.3) reduce to

$$\tau - 2Q\mu = h, \qquad \frac{(A-B)\,\rho_0}{AQ}\left(\frac{\rho_0^2}{A} - \tau\right) + \rho_0\mu = k \tag{3.12}$$

For the variables r, μ not to be constants, it is necessary for the determinant of the system to be equal to zero, from which there follows the relation

$$A = 2B$$

From the system of equations (3.12) we obtain the following relation between h and k:

$$\frac{B\rho_0}{A}\left(\frac{\rho_0^2}{A} - h\right) = Qk \tag{3.13}$$

Equations (2.7), (3.10) and (3.13) give

$$\nu - \rho_0^2 = BQ\left(2\mu - \frac{Ak}{B\rho_0}\right)$$

Differentiating the second equality of (3.1) with respect to t, replacing p, q by their values, taking into account (2.7), (3.13) and substituting the expressions for $\nu - \rho_0^2$, $d\nu/dt$ into (3.11), we obtain the differential equation for the case of Bobylev and Steklov, namely

$$\left(\frac{d\mu}{dt}\right)^2 = \frac{2Q}{B\rho_0}(\rho_0\mu - k)(1-\mu^2) - \frac{(k-\rho_0\mu)^2}{B^2} \tag{3.14}$$

Of the two constants h and k in the integrals (1.3) the solution contains only the constant k [h can be expressed in terms of k by means of (3.13)].

In order to reduce the equation obtained to the form considered by Bobylev, let us make a cyclic permutation of the variables and parameters, replacing p by r, q by p, r by q and the remaining quantities correspondingly. The problem is subject to restrictions which can be put in the form

$$2A = C, \qquad x_0 = y_0 = q = 0, \qquad r = r_0$$

In addition we have $\rho_0 = Cr_0$ $\mu = \gamma_3$. Hence equation (3.14) assumes the form

$$\left(\frac{d\gamma_3}{dt}\right)^2 = \frac{2Mgz_0}{A}\left(\gamma_3 - \frac{k}{Cr_0}\right)\left[1 - \gamma_3^2 - \frac{Cr_0^2}{Mgz_0}\left(\gamma_3 - \frac{k}{Cr_0}\right)\right] \tag{3.15}$$

The constants C_2 and k_b introduced by Bobylev [17] have the expressions

$$C_2 = -\frac{k}{Cr_0}, \qquad k_B = \frac{Ar_0^2}{Mgz_0}$$

Equation (3.14) can be reduced to the form indicated by Steklov [18] if A, x_0, p and γ_1 are replaced by B, y_0, q and γ_2, and the expressions

$$\rho_0\mu - k = \frac{Mgy_0}{q_0}\gamma_1^2, \quad 1-\mu^2 = 1 - \frac{1}{\rho_0^2}\left(k + \frac{Mgy_0}{q_0}\gamma_1^2\right)^2, \quad \frac{d\mu}{dt} = \frac{Mgy_0}{Aq_0^2}\gamma_1\frac{d\gamma_1}{dt}$$

are calculated from the second integral of (1.3) by use of the third Euler equation.

Then equation (3.14) assumes the form

$$\left(\frac{d\gamma_1}{dt}\right)^2 = q_0^2\left[1 - \frac{k^2}{4A^2q_0^2} - \left(1 + \frac{Mgy_0k}{2A^2q_0^3}\right)\gamma_1^2 - \frac{(Mgy_0)^2}{4A^2q_0^4}\gamma_1^4\right] \qquad (3.16)$$

The constants l, n and K introduced by Steklov have the expressions

$$l = \frac{k}{2Aq_0}, \qquad n = \frac{m}{2q_0} = \frac{Mgy_0}{2Aq_0^2}, \qquad K = \frac{k}{A}$$

Consider the seventh and eighth conditions of (3.8). The former in terms of the old variables has the form

$$[(A-B)z_0p + (B-C)x_0r]^2 = L^2 \qquad (L = \text{const}) \qquad (3.17)$$

Formula (3.4) will be written in the form

$$VR_0\sigma = \Lambda R_0\rho + (A-C)x_0z_0L$$

Equalities (3.17) and $\sigma = \sigma_0 = \text{const}$ lead to the condition $\Lambda = 0$, under the assumption that p and r are not constants.

Formula (3.5) will be written in the form

$$BV^2w = V^2\tau - \Lambda V(\nu - B\tau) \pm 2PR_0x_0z_0L\rho + (A-C)UL^2$$

The constants σ_0, w_0 and L can be selected in such a way that this equality is identically satisfied.

The conditions

$$y_0 = 0, \qquad \Lambda = (B-C)x_0^2 - (A-B)z_0^2 = 0 \qquad (3.18)$$

and

$$x_0p + z_0r = \text{const}, \qquad p^2 + q^2 + r^2 = \text{const} \qquad (3.19)$$

taken together give the case of Grioli, i.e. a regular precession about a nonvertical axis.

The meaning of the condition $\Lambda = 0$ is the following. Assuming that $A > B > C$ and that the conditions (3.18) are satisfied, the center of gravity of a heavy rigid body rotating about a fixed point must lie on one of the two perpendiculars to the circular sections of the ellipsoid of inertia constructed with respect to the point of support. In order to prove this, let us construct the central gyration ellipsoid at the center of gravity O and referred to the principal directions, namely

$$\frac{\xi^2}{a_1^2}+\frac{\eta^2}{b_1^2}+\frac{\zeta^2}{c_1^2}=1, \qquad a_1^2>b_1^2>c_1^2 \tag{3.20}$$

where a_1, b_1 and c_1 denote the radii of inertia with respect to the principal central axes.

For the point S, the coordinates of which with respect to these axes are ξ, η, ζ, let the conditions (3.18) be satisfied, where $A = Ma^2$, $B = Mb^2$, $C = Mc^2$ are the moments of inertia of the body with respect to the principal axes of inertia $Sxyz$, intersecting at the point S. Here x_0, y_0 and z_0 are the coordinates of the center of gravity O with respect to the system with the origin at the point S.

In order that the condition $y_0 = 0$ be satisfied, the principal plane xSz, orthogonal to the middle axis of the gyration ellipsoid at the point S

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

must contain the center of gravity O. In addition, this plane must be tangent to the one-sheeted hyperboloid, confocal with the ellipsoid (3.20), passing through the point S and belonging to the system of confocal surfaces

$$\frac{\xi^2}{a_1^2+\lambda_i}+\frac{\eta^2}{b_1^2+\lambda_i}+\frac{\zeta^2}{c_1^2+\lambda_i}=1 \qquad \begin{matrix}(i=1,2,3)\\ \text{for } -b_1^2<\lambda_2<-c_1^2\end{matrix}$$

where

$$\lambda_1=\pi^2-a^2, \quad \lambda_2=\pi^2-b^2, \quad \lambda_3=\pi^2-c^2$$

are the elliptic coordinates of the point S provided

$$\pi^2=\xi^2+\eta^2+\zeta^2$$

holds.

In general, in the coordinate system $O\xi\eta\zeta$, the coordinates of the center of gravity $O(0, 0, 0)$ do not satisfy the equation of the tangent plane to the one-sheeted hyperboloid. This requirement can be met only if the hyperboloid of one sheet degenerates into the outer part of the focal hyperbola

$$\frac{\xi^2}{a_1^2+\lambda_2}+\frac{\zeta^2}{c_1^2+\lambda_2}=1 \qquad (\lambda_2=-b_1^2)$$

i.e. it is necessary that $\eta = 0$. Thus, the point S lies in the coordinate plane $\xi O\zeta$, the point O in the plane xSz, i.e. the axis Sy is parallel to the axis $O\eta$.

From the expression of the Cartesian coordinates of the point S in terms of the elliptic coordinates it follows that $\pi^2 = a_1^2 + b_1^2 + c_1^2 +$

$+ \lambda_1 + \lambda_2 + \lambda_3$. Hence for $\lambda_2 = - b_1^2$, the expressions for the squares of the principal radii of inertia at S are

$$\begin{aligned} a^2 &= a_1^2 + c_1^2 + \lambda_3 \\ b^2 &= a_1^2 + c_1^2 + \lambda_1 - \lambda_2 + \lambda_3 \\ c^2 &= a_1^2 + c_1^2 + \lambda_1 \end{aligned} \tag{3.21}$$

Equating the moments of inertia with respect to the straight line connecting the center of gravity of the body with the pole S, expressed in terms of the radii of inertia with respect to the principal central axes and the principal axes, intersecting at the point S, we obtain in conformity with (3.21)

$$a_1^2\varphi_\zeta^2 + c_1^2\varphi_\xi^2 = -(\lambda_1\varphi_z^2 + \lambda_3\varphi_x^2) \tag{3.22}$$

Here the ϕ's with subscripts denote the direction cosines of the straight line connecting the center of gravity and the pole S with respect to the corresponding axes, assuming that a straight line emanates from the origin of the coordinate system. Eliminating a_1^2 and c_1^2 from the equalities (3.21), we obtain

$$a^2 = b^2 - \lambda_1 + \lambda_2, \qquad c^2 = b^2 + \lambda_2 - \lambda_3$$

The condition $(B - C)x_0^2 - (A - B)z_0^2 = 0$ in this connection gives

$$\lambda_2 = \frac{\lambda_1 z_0^2 + \lambda_3 x_0^2}{\pi^2} = \lambda_1\varphi_z^2 + \lambda_3\varphi_x^2 = -b_1^2$$

Comparison of the equality obtained with that of (3.22) gives

$$b_1^2 = a_1^2\varphi_\zeta^2 + c_1^2\varphi_\xi^2$$

The right-hand side of the equality represents the radius of inertia with respect to the straight line in the plane $\xi O\zeta$ passing through the center of gravity and being perpendicular to the straight line connecting the center of gravity with the pole S. Thus, the locus of points of support S of a heavy rigid body which satisfy the conditions (3.18) is (1) a pair of perpendiculars to the planes of the circular projections of the central gyration ellipsoid, if by the latter planes we understand the planes on which the orthogonal projection of the central gyration ellipsoid is a circle, or (2) a pair of perpendiculars to the planes of the circular sections of the central ellipsoid of inertia.

From the way of constructing the gyration ellipsoid from the ellipsoid of inertia, it follows that in the case when the transformation radius is equal to one, the planes of the circular projections of the central gyration ellipsoid are at the same time the planes of the circular sections of the central ellipsoid of inertia.

Because of the reciprocity of the above mentioned ellipsoids this property is also reciprocal with respect to them.

Substituting the expression $\lambda_2 = \lambda_1\phi_z^2 + \lambda_3\phi_z^2$ into the second equality of (3.21) and taking into account the remaining two equalities, we obtain

$$b^2 = a^2\varphi_z^2 + c^2\varphi_x^2$$

Thus, the disposition of the center of gravity with respect to the point of support is shown.

The center of gravity of a heavy rigid body, rotating about a fixed point and subject to the conditions (3.18), must lie on one of the two perpendiculars to the circular sections of the ellipsoid of inertia, constructed at the point of support.

Guliaev [33] was concerned with the problem of expressing p, q, r, γ_1, γ_2 and γ_3 as explicit functions of time t for the case (3.18) and (3.19).

Consider the particular case when the first constant in (3.19) is equal to zero, i.e. the conditions

$$\begin{aligned} &y_0 = 0, \qquad (B - C)\,x_0^2 - (A - B)\,z_0^2 = 0 \\ &x_0 p + z_0 r = 0, \qquad p^2 + q^2 + r^2 = \text{const} \end{aligned} \tag{3.23}$$

Differentiating the first expression in the second row of (3.23) with respect to time and taking into account the Euler equations (1.1) and the relations of the first row in (3.23), we obtain

$$Ax_0 p + Cz_0 r = -MgR_0^2\frac{\gamma_2}{q} \qquad (R_0^2 = x_0^2 + z_0^2)$$

provided $A \neq C$.

Solving this equation jointly with the first equation from the second row of (3.23), we obtain

$$p = -\frac{MgR_0^2\gamma_2}{(A - C)\,x_0 q}, \qquad r = \frac{MgR_0^2\gamma_2}{(A - C)\,z_0 q}$$

After the substitution into the first and third Eulerian equation of (1.1) and taking into account (3.23), we obtain

$$p = \frac{l}{x_0}, \qquad q = -\frac{MgR_0^2}{(A - C)\,l}\gamma_2, \qquad r = -\frac{l}{z_0} \quad (l = \text{const}) \tag{3.24}$$

The second equation of Euler (1.1) and the second equation of Poisson (1.2), after elimination of $d\gamma_2/dt$, p and r, give an equation which, owing to the first equality of (3.23), can be written in the form

$$(2B - C)\,x_0\gamma_1 + (2B - A)\,z_0\gamma_3 = -\frac{(A - C)^2\,l^2}{MgR_0^2} \tag{3.25}$$

Differentiating this equality with respect to time and substituting for $d\gamma_1/dt$, $d\gamma_3/dt$ their expressions as given by the first and third equation of Poisson (1.2), eliminating p, q and r from them before the substitution by means of (3.24), and assuming that $q \neq 0$, we obtain

$$(2B - A) z_0\gamma_1 - (2B - C) x_0\gamma_3 = -\frac{(A - C) Bl^2}{Mg x_0 z_0} \tag{3.26}$$

Excluding the case $A = C = 2B$, we obtain from (3.25) and (3.26) the expressions

$$\gamma_1 = -\frac{(A - C) l^2}{Mg x_0 R_0^2}, \qquad \gamma_3 = \frac{(A - C) l^2}{Mg z_0 R_0^2} \tag{3.27}$$

After eliminating l, we obtain

$$x_0\gamma_1 + z_0\gamma_3 = 0 \tag{3.28}$$

The second equation of Poisson (1.2), owing to the expression obtained and (3.24), gives γ_2 = const, q = const.

Mlodzeevskii [13] investigated cases when a heavy rigid body, fixed at a point, rotates about a constant (permanent) axis. One such case of motion is when the permanent axis of rotation is vertical and located in the body on a cone of the second order with the vertex at the point of support. This cone under the first condition of (3.23) degenerates into a pair of planes

$$\frac{(A - C) x_0 z_0}{R_0^2} (x_0\gamma_1 + z_0\gamma_3) \gamma_2 = 0$$

The expression (3.28) shows that the motion considered refers to a particular kind of permanent rotations (in the case of Grioli).

The first two classical integrals (1.3) assume the following form

$$h = \frac{Al^2}{x_0^2} + Bq^2 + \frac{Cl^2}{z_0^2}, \qquad k = -\frac{(A - C) l}{MgR_0^2}\left(\frac{Al^2}{x_0^2} + Bq^2 + \frac{Cl^2}{z_0^2}\right)$$

The constants h and k are connected by the relation

$$k = -\frac{(A - C) l}{MgR_0^2} h \tag{3.29}$$

Then from (3.24) and (3.27) the following relations are obtained

$$\gamma_1 = \frac{k}{h} p, \qquad \gamma_2 = \frac{k}{h} q, \qquad \gamma_3 = \frac{k}{h} r \tag{3.30}$$

The third classical integral of (1.1) due to (3.23) and (3.29) gives

$$q^2 = \frac{h^2}{k^2} - \frac{k^2 (MgR_0)^2}{h^2 (A - B)(B - C)}$$

The expressions for p, and r on the basis of the second equality of (3.30) are as follows:

$$p = -\frac{MgR_0^2k}{(A-C)\,x_0h}, \qquad r = \frac{MgR_0^2k}{(A-C)\,z_0h} \tag{3.31}$$

The angular velocity in terms of h and k is given by the expression:

$$\omega^2 = \frac{h^2}{k^2}$$

4. Certain necessary restrictions for particular cases of integrability. Consider the third condition of (3.8) for $V \neq 0$, i.e. $y_0 = z_0 = 0$, $x_0 = R_0$.

The center of gravity of the body lies on the principal axis of inertia, the moment of inertia with respect to which is equal to A.

Here the basic differential equations (2.10) can be written in the form

$$\left(\frac{d\nu}{dt}\right)^2 = -\nu(\tau-h)^2 + 4Q\,[Q(\nu-\rho^2) + k\rho(\tau-h) - Qk^2]$$

$$A^2BC\left(\frac{d\rho}{dt}\right)^2 = -A^2\nu^2 + AJ\nu\rho^2 + A^2B_1\nu\tau + P'\rho^4 + AN\rho^2\tau - A^2BC\tau^2$$

$$(\rho^2 - A\tau)\frac{d\nu}{dt} + A\,[\rho(\tau-h) - 2Qk]\frac{d\rho}{dt} + A(\nu-\rho^2)\frac{d\tau}{dt} = 0 \tag{4.1}$$

where

$$B_1 = B + C, \qquad J = 2A - B - C, \qquad N = 2BC - AB - AC$$

$$P' = (A-B)(C-A), \quad Q = Mgx_0$$

Let us apply Kovalevskaia's method to the system of differential equations obtained.

Assume that formally this system can be satisfied by series of the form

$$\nu = \frac{1}{t^{n_1}}(\nu_0 + \nu_1 t + \nu_2 t^2 + \ldots)$$

$$\rho = \frac{1}{t^{n_2}}(\rho_0 + \rho_1 t + \rho_2 t^2 + \ldots) \tag{4.2}$$

$$\tau = \frac{1}{t^{n_3}}(\tau_0 + \tau_1 t + \tau_2 t^2 + \ldots)$$

where n_1, n_2 and n_3 are positive integers, some of which might be equal to zero, and ν_0, ρ_0 and τ_0 are all different from zero.

For the series (4.3) to represent a particular solution of the system (4.1), they must converge in a certain domain of the variable t. If the particular solution contains at least one essential arbitrary constant (besides the arbitrary constant t_0, assigned to t, and the arbitrary constants h and k), then the series (4.2) will contain at least one essential arbitrary constant among their coefficients.

Consider certain cases for the system of values n_1, n_2 and n_3 when the series (4.2) satisfy the system of differential equations (4.1).

Assume that $n_1 = n_3 = 2$, $n_2 = 1$ $(A \neq B,\ A \neq C)$ and substitute the series (4.2) into the system (4.1).

Equating the coefficients of equal powers of t on the left- and right-hand sides of the equations, we obtain the following systems of algebraic equations for the determination of the coefficients of the series (4.2):

1. $$4\nu_0 + \tau_0^2 = 0 \tag{4.3}$$
$$A\nu_0(\Theta_0 - A\nu_0) - \rho_0^2(P'\rho_0^2 + AN\tau_0 - A^2BC) + A^2BC\tau_0^2 = 0$$
$$2\nu_0 - A\tau_0 = 0$$

2. $$\nu_1(4\nu_0 + \tau_0^2) + 2\tau_1\nu_0\tau_0 = 0$$
$$A\nu_1\Theta_0 - 2\rho_1\Phi_0' - A\tau_1\Psi_0 = 0$$
$$\nu_1(\rho_0^2 + A\tau_0) + \rho_1(4\nu_0 - 3A\tau_0)\rho_0 - A\tau_1\nu_0 = 0$$

3. $$\nu_2\tau_0^2 + 2(\tau_2 - h)\nu_0\tau_0 + \nu_1(\nu_1 + 2\tau_1\tau_0) + \tau_1^2\nu_0 = 0$$
$$A\nu_2\Theta_0 - 2\rho_2(\Phi_0' + A^2BC)\rho_0 - A\tau_2\Psi_0 + A\nu_1(\Theta_1 - A\nu_1) - \rho_1^2\Phi_0 - 2AN\rho_1\tau_1\rho_0 + A^2BC\tau_1^2 = 0$$
$$2A\nu_2\tau_0 + 4\rho_2(\nu_0 - A\tau_0) - 2A\tau_2\nu_0 + A(\tau_2 - h)\rho_0^2 + 2\nu_1\rho_1\rho_0 + 2\rho_1^2(\nu_0 - A\tau_0) - A\rho_1\tau_1\rho_0 = 0$$

4. $$\nu_3(-4\nu_0 + \tau_0^2) + 2[\tau_3\nu_0\tau_0 + \nu_2\tau_1\tau_0 + (\tau_2 - h)(\nu_1\tau_0 + \tau_1\nu_0)] + \nu_1\tau_1^2 - 4Qk\rho_0\tau_0 = 0$$
$$A\nu_3\Theta_0 - 2\rho_3(\Phi_0' + 2A^2BC)\rho_0 - A\tau_3\Psi_0 + A\nu_2\Theta_1 - 2\rho_2\Phi - A\tau_2\Psi_1 - \rho_1^2(\Phi_1 - 2P'\rho_1\rho_0) = 0$$
$$\nu_3(\rho_0^2 - 3A\tau_0) - \rho_3(4\nu_0 - 5A\tau_0)\rho_0 + A\tau_3(3\nu_0 - 2\rho_0^2) - A\nu_2\tau_1 - \rho_2[2\nu_1\rho_0 + \rho_1(4\nu_0 - 5A\tau_0) - 2A\tau_1\rho_0] + A\tau_2\nu_1 - A(\tau_2 - h)\rho_1\rho_0 - \rho_1^2(\nu_1 - A\tau_1) + 2AQk\rho_0 = 0$$

5. $$\nu_4(-8\nu_0 + \tau_0^2) + 2[\tau_4\nu_0\tau_0 - \nu_3(\nu_1 - \tau_1\tau_0) + \tau_3(\nu_1\tau_0 + \tau_1\nu_0)] + \nu_2[2(\tau_2 - h)\tau_0 + \tau_1^2] + (\tau_2 - h)^2\nu_0 + 2(\tau_2 - h)\nu_1\tau_1 - 4Q[k(\rho_1\tau_0 + \tau_1\rho_0) + Q(\nu_0 - \rho_0^2)] = 0$$
$$A\nu_4\Theta_0 - 2\rho_4(\Phi_0' + 3A^2BC)\rho_0 - A\tau_4\Psi_0 + A\nu_3\Theta_1 - 2\rho_3\Phi - A\tau_3\Psi_1 + A\nu_2(\Theta_2 - A\nu_2) - \rho_2^2(\Phi_0 - A^2BC) - 2\rho_2(AN\tau_2\rho_2 + \rho_1\Phi_1) + A^2BC\tau_2^2 - AN\tau_2\rho_1^2 - P'\rho_1^4 = 0$$
$$\nu_4(2\rho_0^2 - 4A\tau_0) - 2\rho_4(2\nu_0 - 3A\tau_0)\rho_0 + A\tau_4(4\nu_0 - 3\rho_0^2) - 2\nu_3(\rho_1\rho_0 - A\tau_1) - \rho_3[2\nu_1\rho_0 + 2\rho_1(2\nu_0 - 3A\tau_0) - 3A\tau_1\rho_0] + A\tau_3(2\nu_1 - 3\rho_1\rho_0) - \rho_2^2(2\nu_0 - 3A\tau_0) - \rho_2\rho_1(2\nu_1 - 3A\tau_1) = 0$$

$$6.\quad \nu_5(-12\nu_0+\tau_0^2)+2[\tau_5\nu_0\tau_0-\nu_4(2\nu_1-\tau_1\tau_0)+\tau_4(\nu_1\tau_0+\tau_1\nu_0)]+$$
$$+\nu_3[2(\tau_2-h)\tau_0+\tau_1^2]+2\{\tau_3[\nu_2\tau_0+(\tau_2-h)\nu_0+\nu_1\tau_1]+\nu_2(\tau_2-h)\tau_1-$$
$$-2Qk\rho_2\tau_0\}+(\tau_2-h)^2\nu_1-4Q[k(\tau_2-h)\rho_0+Q\nu_1+\rho_1(k\tau_1-2Q\rho_0)]=0$$

$$A\nu_5\Theta_0-2\rho_5(\Phi_0'+4A^2BC)\rho_0-A\tau_5\Psi_0+A\tau_4\Theta_1-2\rho_4\Phi-A\tau_4\Psi_1+$$
$$+A\nu_3\Theta_2-2\rho_3[AJ\nu_2\rho_0+\rho_2(\Phi_0-2A^2BC)+AN\tau_2\rho_0+\rho_1\Phi_1]-$$
$$-A\tau_3\Psi_2-2AJ\nu_2\rho_2\rho_1-\rho_2^2(\Phi_1+6P'\rho_1\rho_0)-2AN\rho_2\tau_2\rho_1-4P'\rho_2\rho_1^3=0$$

$$\nu_5(3\rho_0^2-5A\tau_0)-\rho_5(4\nu_0-7A\tau_0)\rho_0+A\tau_5(5\nu_0-4\rho_0^2)+$$
$$+\nu_4(4\rho_1\rho_0-3A\tau_1)-\rho_4[2\nu_1\rho_0+\rho_1(4\nu_0-7A\tau_0)-4A\tau_1\rho_0]+$$
$$+A\tau_4(3\nu_1-5\rho_1\rho_0)+\nu_3(2\rho_2\rho_0-A\tau_2+\rho_1^2)-$$
$$-\rho_3[\rho_2(4\nu_0-7A\tau_0)-A(\tau_2-h)\rho_0+2\nu_1\rho_1-4A\rho_1\tau_1]+$$
$$+A\tau_3(\nu_2-2\rho_2\rho_0-\rho_1^2)-\rho_2^2(\nu_1-2A\tau_1)+A\rho_2[(\tau_2-h)\rho_1-2Qk]=0$$

. . . .

Here for brevity we have set

$$\Theta_0=2A\nu_0-J\rho_0^2-AB_1\tau_0$$
$$\Theta_1=2A\nu_1-J(\rho_0\rho_1+\rho_1\rho_0)-AB_1\tau_1$$
$$\Theta_2=2A\nu_2-J(\rho_0\rho_2+\rho_1\rho_1+\rho_2\rho_0)-AB_1\tau_2$$

$$\Phi_0'=AJ\nu_0+2P'\rho_0^2+AN\tau_0$$
$$\Phi_0=AJ\nu_0+6P'\rho_0^2+AN\tau_0$$
$$\Phi_1=AJ\nu_1+6P'\rho_1\rho_0+AN\tau_0$$
$$\Phi=AJ\nu_1\rho_0+\rho_1\Phi_0+AN\tau_1\rho_0$$

$$\Psi_0=AB_1\nu_0+N\rho_0^2-2ABC\tau_0$$
$$\Psi_1=AB_1\nu_1+N(\rho_0\rho_1+\rho_1\rho_0)-2ABC\tau_1$$
$$\Psi_2=AB_1\nu_2+N(\rho_0\rho_2+\rho_1\rho_1+\rho_2\rho_0)-2ABC\tau_2$$

The first system of (4.3) has for ν_0 and τ_0 the solution

$$\nu_0=-A^2,\qquad \tau_0=-2A$$

The quantity ρ_0 is determined by the equation

$$(A-B)(A-C)\rho_0^4+A^2[(A-B)(A-C)+(A-2B)(A-2C)]\rho_0^2+$$
$$+A^4(A-2B)(A-2C)=0 \qquad (4.4)$$

From the first equation of the second system it follows that $\tau_1 = 0$. The second and third equations, which are homogeneous with respect to ν_1 and ρ_1, have nonzero solutions under the following condition

$$(A-B)(A-C)\rho_0^4 - A^4(A-2B)(A-2C) = 0$$

Equating the resultant of the polynomials (4.4), and the one just obtained, to zero, we obtain for $\nu_1 \neq 0$, $\rho_1 \neq 0$ a condition which contains Kovalevskaia's case or gives

$$(A-B)(A-C) = (A-2B)(A-2C) \qquad (N+BC=0) \qquad (4.5)$$

If we exclude Kovalevskaia's case from consideration, then the common root of these polynomials will be $\rho_0^{(1,2)} = iA$, where $i = \pm\sqrt{-1}$.

If we assume condition (4.5) to be satisfied, then ν_1 can be considered as an arbitrar constant and ρ_1 has the value

$$\rho_1 = \frac{3}{2\rho_0}\nu_1$$

Eliminate ν_2 from the first and second equations and from the first and third equations of the third system of (4.3). Because of the value $\rho_0 = iA$ for the two equations obtained, their determinant with respect to the variables ρ_2 and τ_2 is equal to zero. Then the independent terms of these equations must be proportional to the coefficients of the variables. This fact leads to the condition

$$\nu_1^2[3A - 2(B+C)] = 0$$

The condition obtained, together with (4.5) for $\nu_1 \neq 0$, gives no new cases except Kovalevskaia's, and forces us to conclude that $\nu_1 = \rho_1 = \tau_1 = 0$.

This corresponds to the following values for ρ_0 obtained from the equation (4.4):

$$\rho_0^{(1,2)} = iA, \qquad \rho_0^{(3,4)} = iA\sqrt{\frac{(A-2B)(A-2C)}{(A-B)(A-C)}}$$

The condition (4.5) is not satisfied.

Let us determine successively the coefficients of the series (4.2) for the root $\rho_0^{(1,2)} = iA$. We have

(1) the third system of (4.3)

$$\nu_2 + A\tau_2 - Ah = 0, \qquad B_1\nu_2 + 2i(N-BC)\rho_2 - 2BC\tau_2 = 0$$

$$4\nu_2 - 4iA\rho_2 - A\tau_2 - Ah = 0 \qquad (4.6)$$

and its solution

$$\nu_2 = \frac{2}{3}Ah, \quad \rho_2 = -\frac{ih}{3}, \quad \tau_2 = \frac{h}{3}$$

(2) the fourth system of (4.3)

$$2\nu_3 + A\tau_3 + 2iQk = 0, \qquad B_1\nu_3 - 2iAB_1\rho_3 - 2BC\tau_3 = 0$$

$$5\nu_3 - 6iA\rho_3 - A\tau_3 + 2iQk = 0 \tag{4.7}$$

and its solution

$$\nu_3 = -iQk, \qquad \rho_3 = -\frac{Qk}{2A}, \qquad \tau_3 = 0$$

(3) the fifth system of (4.3)

$$9\nu_4 + 3A\tau_3 + h^2 = 0$$

$$AB_1\nu_4 - 2iA(AB + BC + CA)\rho_4 - 2ABC\tau_4 - \tfrac{1}{9}Nh^2 = 0 \tag{4.8}$$

$$6\nu_4 - 8iA\rho_4 - A\tau_4 + \tfrac{4}{9}h^2 = 0$$

and its solution

$$\nu_4 = -\frac{h^2}{15}, \qquad \rho_4 = -\frac{ih^2}{45A}, \qquad \tau_4 = -\frac{2h^2}{15A}$$

(4) the sixth system of (4.3)

$$4A\nu_5 + A^2\tau_5 - \tfrac{2}{3}iQkh = 0$$

$$A^2B_1\nu_5 - 2iA^2(AB + AC + 2BC)\rho_5 - 2A^2BC\tau_5 + \tfrac{1}{3}iQNkh = 0 \tag{4.9}$$

$$7A\nu_5 - 10iA^2\rho_5 - A^2\tau_5 - iQkh = 0$$

and its solution

$$\nu_5 = 0, \qquad \rho_5 = -\frac{Qkh}{6A^2}, \qquad \tau_5 = \frac{2iQkh}{3A^2}$$

The coefficient matrix of the system (4.6), which is linear and homogeneous with respect to ν_2, ρ_2, τ_2 and h, has the following minors of order three, obtained by deleting the s-th column:

$$\Delta_1 = 4iA^2\lambda, \qquad \Delta_2 = -2A\lambda, \qquad \Delta_3 = 2iA\lambda, \qquad \Delta_4 = -6iA\lambda$$

where $\lambda = N + BC \neq 0$, since condition (4.5) is not satisfied.

If these minors are not zero, then, by selecting any one of the quantities ν_2, ρ_2, τ_2, h as the independent variable, the remaining ones can be uniquely determined in terms of that quantity.

For only some of the unknowns to be considered as arbitrary constants, it is necessary for the minors of the third order to be equal to zero. For the system (4.6) this requirement cannot be satisfied, since $\lambda \neq 0$.

The matrix of the coefficients of the system (4.7), which is linear and homogeneous with respect ot ν_3, ρ_3, τ_3 and k has the following minors of the third order

$$\Delta_1 = 8QA\lambda, \qquad \Delta_2 = 4iQ\lambda, \qquad \Delta_3 = 0, \qquad \Delta_4 = -8iA\lambda$$

The requirement for the vanishing of these minors gives no new conditions, if we reject the condition $Q = Mgx_0 = 0$, previously obtained.

The matrix of the coefficients of the system (4.8), which is linear and homogeneous with respect to ν_4, ρ_4, r_4 and h^2, has the following minors of the third order

$$\Delta_1 = -2iA^2\lambda, \qquad \Delta_2 = -\frac{2}{3}A\lambda, \qquad \Delta_3 = -4iA\lambda, \qquad \Delta_4 = -30iA^2\lambda$$

In this case the necessary requirements for the vanishing of these minors cannot be satisfied.

The matrix of the coefficients of the system (4.9), which is linear and homogeneous with respect to ν_5, ρ_5, r_5, h, has the following minors of order three:

$$\Delta_1 = 0, \qquad \Delta_2 = 2iQA^3k\lambda, \qquad \Delta_3 = -8QA^3k\lambda, \qquad \Delta_4 = -12iA^5\lambda$$

If $\Delta_2 \neq 0$ and $\Delta_3 \neq 0$, then h can be expressed in terms of k and one of the unknown coefficients ρ_5 or r_5. Thus, a restriction connecting the arbitrary constants h and k cannot be established.

The constants h and k, in general, are independent.

It is therefore necessary that $\Delta_2 = \Delta_3 = 0$, which gives a necessary restriction

$$k = 0 \tag{4.10}$$

for the value of the constant in the area integral.

The system (4.9) can be considered as linear and homogeneous with respect to the unknowns ν_5, ρ_5, r_5 and k, the matrix of which, provided that we replace k by h, has the same minors as before. Then we obtain the necessary restriction

$$h = 0 \tag{4.11}$$

for the value of the energy constant.

Thus the coefficients of the series (4.2) contain no essential arbitary constant.

In general, a similar investigation can be carried out for the s-th system of (4.3), not only for the root $\rho_0^{(1,2)} = iA$, but also for the root

$$\rho_0^{(3,4)} = iA\sqrt{\frac{(A-2B)(A-2C)}{(A-B)(A-C)}}$$

This investigation, however, gave no results that need be mentioned here.

5. Cases of integrability when the center of gravity is located in the plane through the axes of equal moments of inertia. Consider the third condition, and one of the following three conditions, of (3.8), for example the third and the sixth conditions, which give

$$A = B, \qquad A \neq C, \qquad y_0 = z_0 = 0 \tag{5.1}$$

The center of gravity of the body lies in the plane passing through the axes of equal moments of inertia.

The basic differential equations (2.10) assume the form

$$\left(\frac{d\nu}{dt}\right)^2 = -\nu(\tau - h)^2 + 4Q\,[Q(\nu - \rho^2) + k\rho(\tau - h) - Qk^2] \tag{5.2}$$

$$A^2C\left(\frac{d\rho}{dt}\right)^2 = -A\nu^2 + (A - C)\,\nu\rho^2 + A(A + C)\,\nu\tau - A(A - C)\,\rho^2\tau - A^2C\tau^2$$

$$(\rho^2 - A\tau)\frac{d\nu}{dt} + A\,[\rho(\tau - h) - 2Qk]\frac{d\rho}{dt} + A(\nu - \rho^2)\frac{d\tau}{dt} = 0$$

where $Q = Mgx_0$.

The condition (4.5) of the preceding paragraph gives Kovalevskaia's case.

If this condition is not satisfied, i.e. in the case $A = 2C$, on the basis of the results of the preceding paragraph we obtain that either $k = 0$, or $h = 0$. Therefore, in seeking further conditions of integrability, it is necessary either to observe one of these requirements, or to seek other conditions in a general form and then impose these conditions, provided that $A \neq 2C$.

Under the conditions (5.1), there exist for the coefficients n_1, n_2 and n_3 in the series (4.2) other values than those considered in Section 4, for example $n_1 = n_3 = 2$, $n_2 = 3$, for which the series (4.2) will satisfy the system of differential equations (5.2).

The fact that for $A = B$ the function p (in the case under consideration $\rho = Ap$) can have a pole of order three was first observed by Appel'rot and Nekrasov (see Appel'rot's theorem in Section 1).

Substituting the series (4.2) for the above choice of values for n_1, n_2 and n_3, into system (5.2), and equating the coefficients of equal powers of t on the left- and right-hand sides of the equations, we obtain the following system of algebraic equations for determining the coefficients of the series:

$$
\begin{aligned}
1.\quad & 4\nu_0^2 + \nu_0\tau_0^2 + 4Q^2\rho_0^2 = 0 \\
& (A - C)(\nu_0 - A\tau_0) - 9A^2C = 0 \\
& 2\nu_0 + A\tau_0 = 0 \\
2.\quad & \nu_1(4\nu_0 + \tau_0^2) + 8Q^2\rho_1\rho_0 + 2\tau_1\nu_0\tau_0 - 4Qk\rho_0\tau_0 = 0 \\
& (A - C)\nu_1\rho_0 + 6A^2C\rho_1 - A(A - C)\tau_1\rho_0 = 0 \\
& \nu_1\rho_0 + \rho_1(4\nu_0 + A\tau_0) + 2A\tau_1\rho_0 = 0 \\
3.\quad & \nu_2\tau_0^2 + 8Q^2\rho_2\rho_0 + 2(\tau_2 - h)\nu_0\tau_0 + \nu_1(\nu_1 + 2\tau_1\tau_0) + \\
& + 4Q\rho_1(Q\rho_1 - k\tau_0) + \tau_1^2\nu_0 - 4Qk\tau_1\rho_0 = 0 \\
& (A - C)\nu_2\rho_0^2 + 12A^2C\rho_2\rho_0 - A(A - C)\tau_2\rho_0^2 + 2(A - C)\nu_1\rho_1\rho_0 + \\
& + 5A^2C\rho_1^2 - 2A(A - C)\rho_1\tau_1\rho_0 = 0 \\
& 4\rho_2\nu_0\rho_0 + 3A(\tau_2 - h)\rho_0^2 + 2\nu_1\rho_1\rho_0 + 2\rho_1^2\nu_0 + 3A\rho_1\tau_1\rho_0 = 0 \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
s + 1.\quad & \nu_s[-4(s - 2)\nu_0 + \tau_0^2] + 8Q^2\rho_s\rho_0 + 2\tau_s\nu_0\tau_0 = f'_{s-1}(\nu, \rho, \tau) \\
& (A - C)\nu_s\rho_0 + 6sA^2C\rho_s - A(A - C)\tau_s\rho_0 = f''_{s-1}(\nu, \rho, \tau) \\
& (s - 2)\nu_s\rho_0 + \rho_s[-4\nu_0 + (s - 2)A\tau_0] - (s + 1)A\tau_s\rho_0 = f'''_{s-1}(\nu, \rho, \tau) \\
& \qquad (s = 1, 2, 3, \ldots)
\end{aligned}
\tag{5.3}
$$

Here the functions $f'_{s-1}(\nu, \rho, \tau)$, $f''_{s-1}(\nu, \rho, \tau)$, $f'''_{s-1}(\nu, \rho, \tau)$ are polynomials which depend on the unknowns $\nu_1, \ldots, \nu_{s-1}, \rho_1, \ldots, \rho_{s-1}, \tau_1, \ldots, \tau_{s-1}$, the indices of which satisfy the condition $i \leqslant s - 1$.

The first system of (5.3) has the solution

$$
\nu_0 = \frac{3A^2C}{A - C}, \quad \rho_0 = \frac{3A^2C}{Q(A - C)}m, \quad \tau_0 = -\frac{6AC}{A - C} \qquad \left(m = i\sqrt{\frac{A + 2C}{A - C}} \neq 0\right) \tag{5.4}
$$

The second system has the determinant $\Delta \neq 0$ and the following solution

$$
\nu_1 = \frac{3ACk}{(A - C)m}, \qquad \rho_1 = -\frac{3ACk}{2Q(A - C)}, \qquad \tau_1 = 0
$$

The solution of the third system of (5.3) is given by

$$
\nu_2 = \frac{3C(4A - 7C)k^2}{4(A + 2C)(A + 3C)}, \qquad \rho_2 = \frac{3C(4A - 7C)k^2}{8Q(A - C)(A + 3C)m}
$$

$$
\tau_2 = h + \frac{C(3A - 10C)k^2}{2A(A + 2C)(A + 3C)} \qquad \text{etc.}
$$

and so on.

Without carrying out the examination of the matrix of the system (5.3) (as this has been done in the preceding sections), let us call attention to the following.

For the solutions of the system of differential equations (5.2) represented in the form of series of the considered form to depend on arbitrary constants (though the number of arbitrary constants may not be complete), in addition to the constant t_0 simply added to t, it is necessary, for values of ν_0, ρ_0, τ_0 as given by (5.4), for the determinant of the $(s + 1)$st system of (5.3) to vanish for certain positive integral values of $s (s = 1, 2, 3, \ldots)$. Equating to zero the determinant of the $(s + 1)$st system of (5.3) with respect to the quantities ν_s, ρ_s and τ_s, we obtain the necessary condition in the form

$$[s(s-1)-3]A = [s(s-1)+6]C \qquad (s = 3, 4, 5, \ldots) \qquad (5.5)$$

This necessary condition for the existence of single-valued solutions cannot be satisfied for $s = 1$ and $s = 2$ by virtue of the mechanical conditions of the problem $(A > 0,\ C > 0)$.

For $s = 3$ we obtain the condition $A = 4C$. Imposing the requirement $k = 0$ obtained in the preceding paragraph, and taking into account (5.1), we obtain the Goriachev and Chaplygin case.

For $s = 4$ we obtain the condition $A = 2C$. Then, in conformity with Section 4, the requirement $k = 0$ is no longer necessary. Taking into account (5.1), we again obtain Kovalevskaia's case.

For $s = 5$ we obtain $A = 26/27\ C$, for $s = 6$ the condition $A = 4/3\ C$, and so on.

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